

Global solution for a kinetic chemotaxis model with internal dynamics and its fast adaptation limit

Jie Liao*

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Abstract

A nonlinear kinetic chemotaxis model with internal dynamics incorporating signal transduction and adaptation is considered. This paper is concerned with: (i) the global solution for this model, and, (ii) its fast adaptation limit to Othmer-Dunbar-Alt type model. This limit gives some insight to the molecular origin of the chemotaxis behaviour.

First, by using the Schauder fixed point theorem, the global existence of weak solution is proved based on detailed a priori estimates, under some quite general assumptions on the model and the initial data. However, the Schauder fixed point theorem does not provide uniqueness. Therefore, additional analysis is required to be developed to obtain uniqueness.

Next, the fast adaptation limit of this model is derived by extracting a weak convergence subsequence in measure space. For this limit, the first difficulty is to show the concentration effect on the internal state. When the small parameter ε , the adaptation time scale, goes to zero, we prove that the solution converges to a Dirac mass in the internal state variable. Another difficulty is the strong compactness argument on the chemical potential, which is essential for passing the nonlinear kinetic equation to the weak limit.

Key words: kinetic chemotaxis model; internal dynamics; global solution; fast adaptation limit

Mathematics Subject Classification (2010): 35B25; 35Q92; 80A30

1 Introduction

Chemotaxis is a mechanism by which cells or bacteria efficiently and rapidly respond to changes in the chemical composition of their environment, for example approaching chemically favorable environments or avoiding unfavorable ones. This behavior is achieved by two major steps: (i) detection of the signal and (ii) integrate the signals received from receptors that triggers the response. We consider the kinetic chemotaxis model with internal dynamics incorporating signal transduction and adaptation

$$\left\{ \begin{array}{l} \partial_t p + v \cdot \nabla_x p + \partial_m \left[\frac{F(m, S)}{\varepsilon} p \right] = \int_V T(v, v', m) p(x, v', m, t) - T(v', v, m) p(x, v, m, t) dv', \\ -\Delta S + S = n(x, t) := \int_0^\infty \int_V p(x, v, m, t) dv dm, \\ p(x, v, m = 0, t) = 0, \quad p(x, v, m = +\infty, t) = 0, \end{array} \right. \quad (1.1)$$

*Department of Mathematics, East China University of Science and Technology, Shanghai, 200237, P. R. China, Email: liaojie@ecust.edu.cn.

where $p(x, v, m, t)$ denotes the cell density at position $x \in \mathbb{R}^d$, velocity $v \in V$, with internal state $m > 0$ (for example, methylation level) and time t , $S(x, t)$ is the chemical potential. The velocity space V is assumed to be a compact domain in \mathbb{R}^d , typical example being unit ball $\{|v| \leq 1\}$. The parameter ε in the model characterises the adaptation time scale.

This model has been developed in [34, 3, 27], and further elaborated by [15, 16], to connect the aspects of the signal transduction and response to the macroscopic equations by using moment closure techniques. This type of model was also studied in some other references, for example [32, 33, 37], to bridge the molecular level pathway dynamics with the cellular behavior such as cell population level motility in some biological systems. These works made possible the development of predictive agent-based models that include the intracellular signaling pathway dynamics. For instance, it is of great biological interest to understand the molecular origins of chemotactic behavior of *E. coli* by deriving a population-level model based on the underlying signaling pathway dynamics. We further mention that, this model also has close relation to the kinetic theory for active particles [6, 5] (KTAP), which has been applied to model various complex systems in life sciences, for instance biological growing tissues [4, 22], social systems [7], behavioural economy [1] or epidemics with gene mutations [11].

The mathematical study of chemotaxis dated back to as early as the work of Patlak [29] in 1953, and further Keller and Segel derived one of the best studied models in mathematical biology at the macroscopic population level [23, 24, 25]. We refer [20] and references therein on this model. The famous Keller-Segel model has successfully explained chemotactic phenomena in slowing changing environments [36]. On the other hand, in order to understand bacterial behavior at the individual level, the Othmer-Dunbar-Alt kinetic model was proposed by Alt (1980) [2] and Othmer et al. (1988) [28] for the description of the chemotactic movement of cells in the presence of a chemical substance, and the Keller-Segel model can be derived by taking the hydrodynamic limit of kinetic models (see [10, 13, 18, 21] and references therein). In the conventional Othmer-Dunbar-Alt kinetic models, a (biased) velocity jump assumption in the turning kernel is always assumed.

This paper is concerned with the study of the kinetic chemotaxis model with internal dynamics (1.1). Our first goal is to prove existence of global solutions for the Cauchy problem of (1.1) under some quite general assumptions. We assume the initial condition

$$p_0(x, v, m) \geq 0, \quad (1 + \langle x \rangle) p_0(x, v, m) \in L^1(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+), \quad p_0(x, v, m) \in L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+), \quad (1.2)$$

where $\langle x \rangle = \sqrt{1 + x^2}$, and

$$\bar{p}_0(x, v) := \int_0^\infty p_0(x, v, m) dm \in L^\infty(\mathbb{R}_x^d \times V). \quad (1.3)$$

We will use the notation

$$\bar{p}(x, v, t) := \int_0^\infty p(x, v, m, t) dm.$$

The assumptions on $F(m, S)$ and $T(v, v', m)$ are:

- Assumption on the adaptation rate F . There exists a non-negative, non-decreasing continuous function $\Pi(\cdot) \in C(\mathbb{R})$ such that

$$|\partial_m F(m, S)| \leq \Pi(S). \quad (1.4)$$

- Assumption on the turning kernel T . The turning kernel T is positive and uniformly bounded: there exist a positive constant C_T such that

$$0 < T(v, v', m) \leq C_T. \quad (1.5)$$

Under the above assumptions, we prove the global existence of weak solution to the system (1.1), see Theorem 2.2 in Section 2. However, the Schauder fixed point theorem does not provide uniqueness. With further assumptions on the coefficients and initial data, the regularity and uniqueness of the global weak solution are also considered in Section 3.

The second goal is to study the fast adaptation limit as $\varepsilon \rightarrow 0$ of (1.1). For the reasons below (compatible with [32, 33]), we make further assumption on F that

$$\begin{aligned} &\text{There exist positive constants } m_{\pm} \text{ and } m_0(S), m_- < m_0(S) < m_+, \text{ such that} \\ &F(m, S) > 0 \quad \text{for } m < m_0(S), \quad F(m, S) < 0 \quad \text{for } m > m_0(S). \end{aligned} \quad (1.6)$$

And to simplify the proof, we also assume the initial data does not concentrate on large m , i.e.,

$$\int_0^\infty \int_{\mathbb{R}_x^d} \int_V m p_0 dv dx dm < \infty. \quad (1.7)$$

Under the above assumptions, we study in Section 4, as $\varepsilon \rightarrow 0$, how the kinetic chemotaxis model with internal dynamics (1.1) is close to the Othmer-Dunbar-Alt type kinetic model

$$\begin{cases} \partial_t \bar{p} + v \cdot \nabla_x \bar{p} &= \int_V T(v, v', m_0(S)) \bar{p}(x, v', t) - T(v', v, m_0(S)) \bar{p}(x, v, t) dv', \\ -\Delta S + S &= n(x, t) := \int_V \bar{p}(x, v, t) dv. \end{cases}$$

The resulting limit, Theorem 4.3, gives some insight to the molecular origins of the chemotaxis behaviour.

We conclude this introduction by mentioning that, other types of scaling and related limits can be considered. For example, based on both fast adaptation and stiff response, the paper [30] studied how the path-wise gradient of chemotactic signal arises from intra-cellular molecular content. See further remarks in the conclusion section.

2 Global existence of solutions

Existence of solutions for this type of problem has been considered in [14]. See also [9]. Here, we extend the results under more general assumptions. We also give uniform bounds which will be useful for later purpose. The parameter ε does not play a role here thus we take $\varepsilon = 1$ for simplicity and rewrite the original equation for p as

$$\partial_t p + v \cdot \nabla_x p + \partial_m [F(m, S)p] = \int_V T(v, v', m) p(x, v', m, t) - T(v', v, m) p(x, v, m, t) dv', \quad (2.1)$$

coupled with the steady elliptic equation for the chemical signal S

$$-\Delta S + S = n(x, t) := \int_0^\infty \int_V p(x, v, m, t) dv dm, \quad (2.2)$$

and the boundary constraint

$$p(x, v, m = 0, t) = 0, \quad p(x, v, m = +\infty, t) = 0. \quad (2.3)$$

Note that from Equation (2.2), we have

$$S(x, t) = G * n(x, t),$$

where G is the Bessel potential. For later use, we recall some properties of the Bessel potential in below.

Proposition 2.1 (*Properties of Bessel potential [35]*)

(P1) $G(x) \geq 0$, $\|G\|_{L^1(\mathbb{R}^d)} = 1$,

(P2) For any spatial dimension d ,

$$G(x) \in L^q(\mathbb{R}^d), \quad \forall 1 \leq q < \frac{d}{d-2}, \quad (\text{we understand } \frac{d}{d-2} = \infty \text{ when } d = 1, 2),$$

(P3) For any $\alpha \in \mathbb{R}$, $\beta > 0$,

$$\int_{|x|>1} |x|^\alpha G^\beta(x) dx < \infty,$$

(P4) For any spatial dimension d ,

$$\nabla G(x) \in L^1(\mathbb{R}^d).$$

We briefly mention that, for $d = 1$, $G(x) = \frac{1}{2}e^{-|x|}$. For $d = 2$, $G(x)$ has the singularity of $\log|x|$ at $|x| \rightarrow 0$. In the case $d \geq 3$, the only singularity of $G(x)$ at $x = 0$ behaves as

$$G(x) \sim |x|^{-(d-2)} \quad \text{as } |x| \rightarrow 0.$$

Also note that $G(x)$ decreases exponentially as $|x| \rightarrow \infty$, therefore we obtain (P2)-(P4).

2.1 Statement of the result

Theorem 2.2 (*Global existence*) Under the assumption (1.4) on F , (1.5) on T , consider system (2.1)-(2.3) with initial data p_0 satisfying (1.2)-(1.3), there exists a global weak solution (p, S) that

$$\|S(\cdot, t)\|_{L^1 \cap L^\infty(\mathbb{R}_x^d)} \leq \|p_0\|_{L^1} + V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_T t e^{2V_d C_T t}), \quad \forall t > 0, \quad (2.4)$$

$$\|p(\cdot, \cdot, \cdot, t)\|_{L^1 \cap L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} \leq \|p_0\|_{L^1} + \|p_0\|_{L^\infty} (1 + \tilde{C} t e^{\tilde{C} t}), \quad \forall t > 0, \quad (2.5)$$

where V_d is the volume of V , $\tilde{C} > 0$ is a constant depends on the estimate of S , C_T and initial data. Moreover,

$$\|\bar{p}(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V)} \leq \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_T t e^{2V_d C_T t}), \quad \forall t > 0, \quad (2.6)$$

$$\|n(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_T t e^{2V_d C_T t}), \quad \forall t > 0. \quad (2.7)$$

We recall that the global existence result was considered in [14], with more precise examples on T and F . They take the turning kernel T as a product of the turning frequency λ and the kernel K , i.e.,

$$T(v, v', m) = \lambda(m) K(v, v', m),$$

in which the kernel K is non-negative and satisfies the normalisation condition

$$\int_V K(v, v', m) dv = 1,$$

where V is a symmetric compact set in \mathbb{R}^d . The simplest example of K is

$$K(v, v', m) = \frac{1}{V_d},$$

to assume constant turning probability of changing velocity from v' to v , or

$$K(v, v', m) = k\theta, \quad \cos(\theta) = \frac{v \cdot v'}{|v||v'|},$$

to assume that the turning kernel is a function of the angle between original and new velocities. More general, one can assume that it is uniformly bounded by a constant, i.e.,

$$|K(v, v', m)| \leq C.$$

For the (output) turning frequency $\lambda(m)$, which is related to the (input) signal function seen by a cell, to prevent formation of singularities, the authors in [14] suppose the growth estimate

$$\lambda(m) \leq C \left(1 + \Lambda(S) + \left| \frac{\partial S}{\partial t} + v \cdot \frac{\partial S}{\partial x} \right| \right), \quad \forall x, v, t,$$

where $\Lambda \in C(\mathbb{R}^+)$ is a non-negative, non-decreasing continuous function. An assumption that, as we see it below, is not necessary for our mathematical treatment.

2.2 A priori estimates

- L^1 bounds on p , \bar{p} and n .

Take the integration of (2.1) with respect to x, v, m variables, we have the mass conservation

$$\frac{d}{dt} \int_{\mathbb{R}_x^d} \int_0^\infty \int_V p(x, v, m, t) dv dm dx \equiv 0,$$

which is also

$$\begin{aligned} \|p(\cdot, \cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} &= \|\bar{p}(\cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^d \times V)} = \|n(\cdot, t)\|_{L^1(\mathbb{R}_x^d)} \\ &\equiv \int_{\mathbb{R}_x^d} \int_0^\infty \int_V p_0(x, v, m) dv dm dx = \|p_0\|_{L^1}. \end{aligned} \quad (2.8)$$

- L^∞ bounds on \bar{p} and n .

On the other hand, take the integration of (2.1) only with respect to m over $[0, \infty)$ and use the boundary condition (2.3), we have

$$\partial_t \bar{p} + v \cdot \nabla_x \bar{p} = \int_0^\infty \int_V T(v, v', m) p(x, v', m, t) - T(v', v, m) p(x, v, m, t) dv' dm,$$

then it can be further represented by

$$\begin{aligned} \bar{p}(x, v, t) &= \bar{p}_0(x - vt, v) + \int_0^t \int_0^\infty \int_V \{ T(v, v', m) p(x - v(t - s), v', m, s) \\ &\quad - T(v', v, m) p(x - v(t - s), v, m, s) \} dv' dm ds. \end{aligned}$$

Recall the initial condition (1.3) and assumption (1.5) on T , take the $L^\infty(\mathbb{R}_x^d \times V)$ norm on both sides of the above equation to get

$$\|\bar{p}(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V)} \leq \|\bar{p}_0\|_{L^\infty} + 2V_d C_{\mathcal{T}} \int_0^t \|\bar{p}(\cdot, \cdot, s)\|_{L^\infty(\mathbb{R}_x^d \times V)} ds.$$

Then, apply Gronwall's inequality, we readily find (2.6).

Next, from the definition of n , we have that

$$\|n(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq V_d \|\bar{p}(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V)},$$

thus we have (2.7).

- L^1 and L^∞ bounds on S .

Recall that $S(x, t) = G * n(x, t)$, by Young's inequality for convolution, it is direct that

$$\|S(\cdot, t)\|_{L^1(\mathbb{R}_x^d)} \leq \|G\|_{L^1(\mathbb{R}_x^d)} \|n(\cdot, t)\|_{L^1(\mathbb{R}_x^d)} = \|p_0\|_{L^1}, \quad (2.9)$$

where (2.8) is used, and

$$\|S(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq \|G\|_{L^1(\mathbb{R}_x^d)} \|n(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} t e^{2V_d C_{\mathcal{T}} t}),$$

in which (2.7) is used. Then this inequality and (2.9) gives the estimate (2.4).

- L^∞ bound on p .

Now we introduce the characteristics of Equation (2.1) by

$$\frac{d\mathbf{X}}{ds} = v, \quad \frac{d\mathbf{V}}{ds} = 0, \quad \frac{d\mathbf{M}}{ds} = F(\mathbf{M}(s), S(\mathbf{X}(s), s)), \quad (2.10)$$

and along time-backward characteristics starting at (x, v, m, t) , we have for $0 \leq s \leq t$,

$$\mathbf{X}(s) = x - v(t - s), \quad \mathbf{M}(s) = m - \int_s^t F(\mathbf{M}(\tau), S(\mathbf{X}(\tau), \tau)) d\tau,$$

integrate (2.1) along the characteristic from 0 to t we get

$$\begin{aligned} p(x, v, m, t) &= p_0(\mathbf{X}(0), v, \mathbf{M}(0)) - \int_0^t \partial_m F(\mathbf{M}(\tau), S(\mathbf{X}(\tau), \tau)) p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) d\tau \\ &+ \int_0^t \int_V T(v, v', \mathbf{M}(\tau)) p(\mathbf{X}(\tau), v', \mathbf{M}(\tau), \tau) - T(v', v, \mathbf{M}(\tau)) p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) dv' d\tau. \end{aligned} \quad (2.11)$$

Now, take the $L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)$ norm on both sides, and under the assumptions (1.4)-(1.5), we have

$$\begin{aligned} \|p(\cdot, \cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} &\leq \|p_0\|_{L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} \\ &+ [\Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) + 2V_d C_{\mathcal{T}}] \int_0^t \|p(\cdot, \cdot, \cdot, \tau)\|_{L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} d\tau, \end{aligned} \quad (2.12)$$

applying Gronwall's inequality again, we obtain (2.5).

2.3 Proof of Theorem 2.2

We use the Schauder fixed point theorem to prove global existence.

Theorem 2.3 *(The Schauder fixed point theorem [19, 31]) Let X be a normed vector space, and let $K \subset X$ be a non-empty, bounded, and convex set. Then for any given continuous mapping $\phi : K \rightarrow K$, with $\phi(K)$ being pre-compact, there exists a fixed point $x \in K$ such that $\phi(x) = x$.*

To apply this theorem, we separate the presentation into three steps.

Step 1. Setting of the function space and the mapping. Fix $T_0 > 0$, we define

$$X := L^1([0, T_0] \times \mathbb{R}_x^d),$$

and take a bounded convex subset K in X defined by

$$K = \{S \in X \mid S \geq 0, \|S(\cdot, t)\|_{L^1} \leq \|p_0\|_{L^1}, \|S(\cdot, t)\|_{L^\infty} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} T_0 e^{2V_d C_{\mathcal{T}} T_0}), \forall t \in [0, T_0]\}.$$

Then we define the mapping ϕ . We start from a function $S \in K$, and construct $p \in L^\infty([0, T_0], L^1(\mathbb{R}_x^d \times V \times R_m^+))$ according to (2.1) with S fixed. This is possible because it is a linear operator and the characteristics are well defined according to (2.10).

Next, the integration of p with respect to (v, m) defines $n \in L^\infty([0, T_0], L^1(\mathbb{R}_x^d))$. By the above a priori estimate (2.8) and (2.7), we have

$$\|n(\cdot, t)\|_{L^1(\mathbb{R}_x^d)} = \|p_0\|_{L^1}, \quad \forall t \in [0, T_0],$$

$$\|n(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} t e^{2V_d C_{\mathcal{T}} t}), \quad \forall t \in [0, T_0].$$

Note that by interpolation, for any $q \in (1, \infty)$,

$$\|n(\cdot, t)\|_{L^q(\mathbb{R}_x^d)} \leq \|n(\cdot, t)\|_{L^1(\mathbb{R}_x^d)}^{\frac{1}{q}} \|n(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)}^{\frac{q-1}{q}}. \quad (2.13)$$

Further, we construct Σ as the solution to

$$-\Delta \Sigma + \Sigma = n := \int_0^\infty \int_V p(x, v, m, t) dv dm. \quad (2.14)$$

By using Bessel potential and Young's inequality for convolution (as the proof of (2.9)), we have

$$\|\Sigma(\cdot, t)\|_{L^1} \leq \|p_0\|_{L^1}, \quad \|\Sigma(\cdot, t)\|_{L^\infty} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} T_0 e^{2V_d C_{\mathcal{T}} T_0}), \quad \forall t \in [0, T_0],$$

thus it holds that $\Sigma \in K$, therefore we have defined a mapping on K that

$$\phi : S \mapsto \Sigma.$$

Step 2. Continuity. To prove the continuity of the mapping $\phi : S \mapsto \Sigma$, we observe that, firstly, the mapping

$$\phi_1 : p \mapsto \Sigma$$

defined by (2.14) is continuous, because it is a bounded linear operator from L^q to L^q for $1 \leq q \leq \infty$. Secondly, the mapping

$$\phi_2 : S \mapsto p$$

is also continuous, which can be seen exactly from the representation formula (2.11) (and also (2.12)). Indeed, the characteristics are uniquely defined and continuous with respect to parameters, although they are not Lipschitz, neither relevant for DiPerna-Lions theory [12].

In conclusion, the composition $\phi : S \mapsto \Sigma$ is continuous.

Step 3. Compactness of the mapping. To use the Schauder fixed point theorem, we need further to prove that

$$\phi(K) \text{ is pre-compact in } K. \quad (2.15)$$

The proof of (2.15) consists of the following three claims.

Claim 1. Local compactness in space:

$$\Sigma(\cdot, t) \in W^{2,q}(\mathbb{R}_x^d), \quad \forall t \in [0, T_0], \quad \forall 1 < q < \infty.$$

This claim is clear by using standard elliptic regularity estimate [19], since Σ is defined by (2.14) and recall (2.13) that $n(\cdot, t) \in L^q(\mathbb{R}_x^d)$, for all $q > 1$.

Claim 2. Local compactness in time:

$$\partial_t \Sigma(\cdot, t) \in W^{1,q}(\mathbb{R}_x^d), \quad \forall t \in [0, T_0], \quad \forall 1 < q < \infty.$$

This is because, take time derivative of the elliptic equation for Σ we have

$$-\Delta \partial_t \Sigma + \partial_t \Sigma = \partial_t n = \nabla_x \cdot \int_0^\infty \int_V v p dv dm.$$

Note that for any t , the integration on the right hand side of above is in $L^q(\mathbb{R}_x^d)$, for any $q > 1$, the standard elliptic regularity theory shows that $\partial_t \Sigma \in W^{1,q}(\mathbb{R}_x^d)$, and actually

$$\begin{aligned} \|\partial_t \Sigma(\cdot, t)\|_{W^{1,q}(\mathbb{R}_x^d)} &\leq \bar{C} \left\| \int_0^\infty \int_V v p dv dm \right\|_{L^q(\mathbb{R}_x^d)} \leq \bar{C} \text{diam}(V) \left\| \int_V \bar{p} dv \right\|_{L^q(\mathbb{R}_x^d)} \\ &\leq \tilde{C} \left\| \int_V \bar{p} dv \right\|_{L^1 \cap L^\infty(\mathbb{R}_x^d)} \leq \tilde{C} \|\bar{p}\|_{L^1 \cap L^\infty(\mathbb{R}_x^d \times V)}, \quad \forall t \in [0, T_0], \end{aligned}$$

where \bar{C} independent of S is a constant given by elliptic estimate and \tilde{C} is a genetic constant which differs from line to line.

Claim 3. Control for $|x| \sim \infty$:

$$\int_{\mathbb{R}_x^d} \langle x \rangle \Sigma dx < \infty, \quad \forall t \in [0, T_0], \quad \text{where} \quad \langle x \rangle := \sqrt{1 + |x|^2}. \quad (2.16)$$

For this, note that Σ satisfies (2.14), then

$$\int_{\mathbb{R}_x^d} \langle x \rangle \Sigma dx = \int_{\mathbb{R}_x^d} \langle x \rangle n dx + \int_{\mathbb{R}_x^d} \langle x \rangle \Delta \Sigma dx, \quad (2.17)$$

in which the second term is bounded because

$$\int_{\mathbb{R}_x^d} \langle x \rangle \cdot \Delta \Sigma dx = \int_{\mathbb{R}_x^d} \Delta \langle x \rangle \cdot \Sigma dx = \int_{\mathbb{R}_x^d} \left(\frac{d-1}{\langle x \rangle} + \frac{1}{\langle x \rangle^3} \right) \cdot \Sigma dx \leq d \|\Sigma\|_{L^1(\mathbb{R}_x^d)}.$$

To control the first term on the right hand side of (2.17), we multiply Equation (2.1) by $\langle x \rangle$ and integrate:

$$\frac{d}{dt} \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \langle x \rangle p dv dm dx + \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \langle x \rangle v \cdot \nabla_x p dv dm dx = 0.$$

Then

$$\frac{d}{dt} \int_{\mathbb{R}_x^d} \langle x \rangle n dx = \frac{d}{dt} \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \langle x \rangle p dv dm dx = \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \frac{v \cdot x}{\langle x \rangle} p dv dm dx \leq \|p_0\|_{L^1}.$$

Therefore from the initial date (1.2) we have the bound

$$\int_{\mathbb{R}_x^d} \langle x \rangle n dx \leq \int_{\mathbb{R}_x^d} \langle x \rangle n_0 dx + t \|p_0\|_{L^1} = \|\langle x \rangle p_0\|_{L^1} + t \|p_0\|_{L^1}, \quad \forall t \in [0, T_0],$$

then we have proved (2.16), and further,

$$\langle x \rangle \Sigma \in X = L^1([0, T_0] \times \mathbb{R}_x^d). \quad (2.18)$$

The property (2.18) yields:

$$\forall \varepsilon_0 > 0, \exists \Omega \subset [0, T_0] \times \mathbb{R}_x^d, \text{ bounded, measurable, such that } \|\Sigma\|_{L^1([0, T_0] \times \mathbb{R}_x^d \setminus \Omega)} < \varepsilon_0. \quad (2.19)$$

This is because

$$\int_0^{T_0} \int_{|x| \geq R} \Sigma dx dt \leq \frac{1}{\langle R \rangle} \int_0^{T_0} \int_{|x| \geq R} \langle x \rangle \Sigma dx dt \rightarrow 0 \text{ as } R \rightarrow \infty.$$

In conclusion, for any fixed Ω with finite measure defined in (2.19), by Claim 1 and Claim 2, we have

$$\{\Sigma|_\Omega\} \text{ is precompact in } L^1(\Omega),$$

together with (2.19), the weighted in space control away from Ω , we proved (2.15), by using strong compactness criterion in Brezis [8] (Theorem 4.26, Corollary 4.27, p. 111).

Now we use the Schauder Theorem 2.3 to get a fixed point which proves existence of solution $S \in K \subset X = L^1([0, T_0] \times \mathbb{R}_x^d)$ for all $T_0 > 0$. And next, we can also construct the global solution $p \in L^\infty([0, T_0], L^1 \cap L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+))$ for all $T_0 > 0$ by the above a priori estimates. The proof of Theorem 2.2 is complete. \square

3 Regularity and uniqueness of the solution

In above section, the existence of global weak solution is derived by using the Schauder fixed point theorem. However, the Schauder fixed point theorem does not provide uniqueness. Therefore, additional analysis is required to be developed to obtain uniqueness.

Note the property (P4) of G in Proposition 2.1 and the a priori estimate (2.7), by using Young's inequality we have

$$\|\partial_{x_i} S\|_{L^\infty(\mathbb{R}_x^d)} \leq \|\partial_{x_i} G\|_{L^1(\mathbb{R}_x^d)} \|n\|_{L^\infty(\mathbb{R}_x^d)} < \infty. \quad (3.1)$$

Also recall (2.4) that $S(\cdot, t) \in L^\infty(\mathbb{R}_x^d)$, then we obtain

Proposition 3.1 *Let S be the weak solution of (2.1)-(2.3) defined in Theorem 2.2. Then*

$$S(\cdot, t) \in W^{1,\infty}(\mathbb{R}_x^d).$$

By further assumptions on regularity of coefficients and initial data, we can get higher regularity of p . For this purpose, we further assume

- There exists a non-negative, non-decreasing continuous function $\tilde{\Pi}(\cdot) \in C(\mathbb{R})$ such that

$$|\partial_S F(m, S)| + |\partial_{mm} F(m, S)| + |\partial_m \partial_S F(m, S)| \leq \tilde{\Pi}(S). \quad (3.2)$$

- There exist a positive constant $\tilde{C}_{\mathcal{T}}$ such that

$$|\nabla_v T(v, v', m)| + |\nabla_{v'} T(v, v', m)| + |\partial_m T(v, v', m)| \leq \tilde{C}_{\mathcal{T}}. \quad (3.3)$$

Proposition 3.2 *Under the assumptions on Theorem 2.2. Further assume (3.2)-(3.3) and let initial data $p_0 \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)$. Then the weak solution of (2.1)-(2.3) satisfies*

$$p(\cdot, \cdot, \cdot, t) \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+). \quad (3.4)$$

Proof. 1. *Regularity on x .* Differentiate (2.1) with respect to x_i , $1 \leq i \leq d$, we have

$$\begin{aligned} \partial_t \partial_{x_i} p + v \cdot \nabla_x \partial_{x_i} p + F(m, S) \partial_m \partial_{x_i} p &= -2 \partial_m F(m, S) \partial_{x_i} p - \partial_m \partial_S F(m, S) \partial_{x_i} S p \\ &+ \int_V T(v, v', m) \partial_{x_i} p(x, v', m, t) - T(v', v, m) \partial_{x_i} p(x, v, m, t) dv'. \end{aligned}$$

Integrate along the characteristics defined by (2.10) from 0 to t :

$$\begin{aligned} \partial_{x_i} p(x, v, m, t) &= \partial_{x_i} p_0(\mathbf{X}(0), v, \mathbf{M}(0)) - \int_0^t 2 \partial_m F(\mathbf{M}(\tau), S(\mathbf{X}(\tau), \tau)) \partial_{x_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) d\tau \\ &- \int_0^t \partial_m \partial_S F(\mathbf{M}(\tau), S(\mathbf{X}(\tau), \tau)) \partial_{x_i} S(\mathbf{X}(\tau), \tau) p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) d\tau \\ &+ \int_0^t \int_V T(v, v', \mathbf{M}(\tau)) \partial_{x_i} p(\mathbf{X}(\tau), v', \mathbf{M}(\tau), \tau) - T(v', v, \mathbf{M}(\tau)) \partial_{x_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) dv' d\tau. \end{aligned}$$

Using the assumptions on the coefficients and initial data as stated above, we get

$$\begin{aligned} |\partial_{x_i} p(x, v, m, t)| &\leq |\partial_{x_i} p_0(\mathbf{X}(0), v, \mathbf{M}(0))| + 2\Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) \int_0^t |\partial_{x_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ \tilde{\Pi}(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) \|\partial_{x_i} S\|_{L^\infty(\mathbb{R}_x^d)} \int_0^t |p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ 2V_d C_{\mathcal{T}} \int_0^t |\partial_{x_i} p(\mathbf{X}(\tau), v', \mathbf{M}(\tau), \tau)| d\tau, \end{aligned}$$

that is,

$$\begin{aligned} |\partial_{x_i} p(x, v, m, t)| &\leq |\partial_{x_i} p_0(\mathbf{X}(0), v, \mathbf{M}(0))| \\ &+ 2[\Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) + V_d C_{\mathcal{T}}] \int_0^t |\partial_{x_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ \tilde{\Pi}(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) \|\partial_{x_i} S\|_{L^\infty(\mathbb{R}_x^d)} \int_0^t |p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau. \end{aligned} \quad (3.5)$$

2. *Regularity on v .* Differentiate (2.1) with respect to v_i , $1 \leq i \leq d$, we have

$$\begin{aligned} \partial_t \partial_{v_i} p + v \cdot \nabla_x \partial_{v_i} p + F(m, S) \partial_m \partial_{v_i} p &= -\partial_m F(m, S) \partial_{v_i} p - \partial_{x_i} p \\ &+ \int_V T(v, v', m) \partial_{v_i} p(x, v', m, t) - T(v', v, m) \partial_{v_i} p(x, v, m, t) dv' \\ &+ \int_V \partial_{v_i} T(v, v', m) p(x, v', m, t) - \partial_{v_i} T(v', v, m) p(x, v, m, t) dv'. \end{aligned}$$

Similarly, integrate along the characteristics defined by (2.10) from 0 to t , and use the assumptions, we obtain

$$\begin{aligned} |\partial_{v_i} p(x, v, m, t)| &\leq |\partial_{v_i} p_0(\mathbf{X}(0), v, \mathbf{M}(0))| + \int_0^t |\partial_{x_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ \Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) \int_0^t |\partial_{v_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ 2V_d \tilde{C}_\tau \int_0^t |p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau + V_d C_\tau \int_0^t |\partial_{v_i} p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau. \end{aligned} \quad (3.6)$$

3. *Regularity on m .* Next, differentiate (2.1) with respect to m , we have

$$\begin{aligned} \partial_t \partial_m p + v \cdot \nabla_x \partial_m p + F(m, S) \partial_m \partial_m p &= -\partial_{mm} F(m, S) p - 2\partial_m F(m, S) \partial_m p \\ &+ \int_V T(v, v', m) \partial_m p(x, v', m, t) - T(v', v, m) \partial_m p(x, v, m, t) dv' \\ &+ \int_V T_m(v, v', m) p(x, v', m, t) - T_m(v', v, m) p(x, v, m, t) dv'. \end{aligned}$$

Integrate along the characteristics defined by (2.10) from 0 to t , we obtain

$$\begin{aligned} |\partial_m p(x, v, m, t)| &\leq |\partial_m p_0(\mathbf{X}(0), v, \mathbf{M}(0))| \\ &+ [2\Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) + 2V_d C_\tau] \int_0^t |\partial_m p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau \\ &+ [\tilde{\Pi}(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) + 2V_d \tilde{C}_\tau] \int_0^t |p(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau)| d\tau. \end{aligned} \quad (3.7)$$

In summary, recall the estimate (2.5) for p and Proposition 3.1 for S , combining (3.5)-(3.7) and taking L^q norms, using Gronwall's inequality, we deduce, for $1 \leq q \leq \infty$, $\forall t > 0$,

$$\|\nabla_x p(\cdot, \cdot, \cdot, t)\|_{L^q} + \|\nabla_v p(\cdot, \cdot, \cdot, t)\|_{L^q} + \|\partial_m p(\cdot, \cdot, \cdot, t)\|_{L^q} \leq C(\|S(\cdot, t)\|_{W^{1,\infty}}, \|p_0\|_{W^{1,q}}, V_d, C_\tau, \tilde{C}_\tau) < \infty.$$

Together with (2.5), we have (3.4). \square

For later use, we derive the following a priori estimates.

Proposition 3.3 *Under the assumptions on Proposition 3.2, and further assume the initial data satisfy $\partial_m p_0 \in L_x^\infty(L_{v,m}^1)$. Then for all $t > 0$,*

$$\|p\|_{L_x^\infty(L_{v,m}^1)} + \|\partial_m p\|_{L_x^\infty(L_{v,m}^1)} < \infty.$$

Proof. Firstly, recall the a priori estimate (2.6) on \bar{p} thus $p \in L_{x,v}^\infty(L_m^1)$. Note V is compact, then we readily have

$$\|p\|_{L_x^\infty(L_{v,m}^1)} < \infty.$$

Next we prove $p_m \in L_x^\infty(L_{v,m}^1)$. For this, we take $L_x^\infty(L_{v,m}^1)$ norm on both sides of (3.7), then the result follows by using again Gronwall's inequality. \square

We now state the uniqueness result.

Proposition 3.4 *Under the assumptions on Proposition 3.3, the weak solution of (2.1)-(2.3) is unique.*

Proof. We follow the steps in [26]. Assume both (p_1, S_1) and (p_2, S_2) be weak solutions of (2.1)-(2.3) with same initial data p_0 satisfying (1.2)-(1.3). Denote

$$\tilde{p} = p_1 - p_2, \quad \tilde{S} = S_1 - S_2.$$

Then we have

$$\begin{aligned} & \partial_t \tilde{p} + v \cdot \nabla_x \tilde{p} + \partial_m [F(m, S_1) \tilde{p} + (F(m, S_1) - F(m, S_2)) p_2] \\ &= \int_V T(v, v', m) \tilde{p}(x, v', m, t) - T(v', v, m) \tilde{p}(x, v, m, t) dv', \end{aligned} \quad (3.8)$$

$$- \Delta \tilde{S} + \tilde{S} = \tilde{n}(x, t) := \int_0^\infty \int_V \tilde{p}(x, v, m, t) dv dm, \quad (3.9)$$

and the initial data

$$\tilde{p} = 0. \quad (3.10)$$

From (3.9) we write $\tilde{S} = G * \tilde{n}$. By Young's inequality,

$$\|\tilde{S}\|_{L_x^1} \leq \|\tilde{p}(t)\|_{L_{x,v,m}^1}. \quad (3.11)$$

Next, from (3.8) we have

$$\begin{aligned} & \partial_t \tilde{p} + v \cdot \nabla_x \tilde{p} + F(m, S_1) \partial_m \tilde{p} \\ &= -\partial_m F(m, S_1) \tilde{p} - \partial_m \left(p_2 \int_0^1 \partial_S F(m, (1-\theta)S_2 + \theta S_1) d\theta \right) \tilde{S} \\ &+ \int_V T(v, v', m) \tilde{p}(x, v', m, t) - T(v', v, m) \tilde{p}(x, v, m, t) dv'. \end{aligned}$$

Integrate along the characteristics defined by (2.10) from 0 to t , with S replaced by S_1 , we get

$$\begin{aligned} \tilde{p}(x, v, m, t) &= \tilde{p}_0(\mathbf{X}(0), v, \mathbf{M}(0)) - \int_0^t \partial_m F(\mathbf{M}(\tau), S_1(\mathbf{X}(\tau), \tau)) \tilde{p}(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) d\tau \\ &- \int_0^t \partial_m \left(p_2 \int_0^1 \partial_S F(m, (1-\theta)S_2 + \theta S_1) d\theta \right) (\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) \tilde{S}(\mathbf{X}(\tau), \tau) d\tau \\ &+ \int_0^t \int_V T(v, v', \mathbf{M}(\tau)) \tilde{p}(\mathbf{X}(\tau), v', \mathbf{M}(\tau), \tau) - T(v', v, \mathbf{M}(\tau)) \tilde{p}(\mathbf{X}(\tau), v, \mathbf{M}(\tau), \tau) dv' d\tau. \end{aligned}$$

Take $L_{x,v,m}^1$ norm on both sides, we obtain

$$\begin{aligned} \|\tilde{p}(t)\|_{L_{x,v,m}^1} &\leq \|\tilde{p}_0\|_{L_{x,v,m}^1} + [\Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) + 2V_d C_{\mathcal{T}}] \int_0^t \|\tilde{p}(s)\|_{L_{x,v,m}^1} ds \\ &\quad + \left(\|p\|_{L_x^\infty(L_{v,m}^1)} + \|p_m\|_{L_x^\infty(L_{v,m}^1)} \right) \Pi(\max_{0 \leq s \leq t} |S(\mathbf{X}(s), s)|) \|\tilde{S}\|_{L_x^1}. \end{aligned}$$

Notice from Proposition 3.3 that $p, p_m \in L_x^\infty(L_{v,m}^1)$, and use (3.11), we get

$$\|\tilde{p}(t)\|_{L_{x,v,m}^1} \leq \|\tilde{p}_0\|_{L_{x,v,m}^1} + C(t) \int_0^t \|\tilde{p}(s)\|_{L_{x,v,m}^1} ds,$$

where $C(t)$ is bounded for any given t . Recall the zero initial data (3.10), thus by applying Gronwall's inequality we have

$$\|\tilde{p}(t)\|_{L_{x,v,m}^1} \equiv 0,$$

then we have proved the uniqueness of the global weak solution. \square

4 Fast adaptation limit

In this section we investigate, as $\varepsilon \rightarrow 0$, the limiting behavior of the system

$$\left\{ \begin{aligned} \partial_t p^\varepsilon + v \cdot \nabla_x p^\varepsilon + \partial_m \left[\frac{F(m, S^\varepsilon)}{\varepsilon} p^\varepsilon \right] &= \int_V T(v, v', m) p^\varepsilon(x, v', m, t) - T(v', v, m) p^\varepsilon(x, v, m, t) dv' \\ -\Delta S^\varepsilon + S^\varepsilon &= n^\varepsilon(x, t) := \int_0^\infty \int_V p^\varepsilon(x, v, m, t) dv dm, \\ p^\varepsilon(x, v, m = 0, t) &= 0, \quad p^\varepsilon(x, v, m = +\infty, t) = 0, \end{aligned} \right. \quad (4.1)$$

with initial data p_0 which satisfies (1.2)-(1.3). Below we denote $\mathcal{M}(\Omega)$ the space of Radon measures on Ω , C_0 is the Banach space of continuous functions which vanishes at ∞ , and the notation

$$\bar{p}^\varepsilon(x, v, t) := \int_0^\infty p^\varepsilon(x, v, m, t) dm.$$

First of all, we recall the a priori estimates including the parameter ε , which is

Proposition 4.1 *Under the same assumptions in Theorem 2.2. The solution to (4.1) satisfies*

$$\|p^\varepsilon(\cdot, \cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} \leq \|p_0\|_{L^\infty} \left(1 + \frac{\tilde{C}t}{\varepsilon} e^{\frac{\tilde{C}t}{\varepsilon}}\right), \quad \forall t > 0, \quad (4.2)$$

and

$$\|p^\varepsilon(\cdot, \cdot, \cdot, t)\|_{L^1(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+)} = \|p_0\|_{L^1}, \quad \forall t > 0, \quad (4.3)$$

$$\|S^\varepsilon(\cdot, t)\|_{L^1 \cap L^\infty(\mathbb{R}_x^d)} \leq \|p_0\|_{L^1} + V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} t e^{2V_d C_{\mathcal{T}} t}), \quad \forall t > 0, \quad (4.4)$$

$$\|n^\varepsilon(\cdot, t)\|_{L^\infty(\mathbb{R}_x^d)} \leq V_d \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} t e^{2V_d C_{\mathcal{T}} t}), \quad \forall t > 0, \quad (4.5)$$

$$\|\bar{p}^\varepsilon(\cdot, \cdot, t)\|_{L^\infty(\mathbb{R}_x^d \times V)} \leq \|\bar{p}_0\|_{L^\infty} (1 + 2V_d C_{\mathcal{T}} t e^{2V_d C_{\mathcal{T}} t}), \quad \forall t > 0. \quad (4.6)$$

From the above proposition, we have

Lemma 4.2 *Let $(p^\varepsilon, S^\varepsilon)$ be solution of (4.1) and further assume (1.6)-(1.7). Fix any $T > 0$. Then, up to a subsequence, we have*

$$p^\varepsilon \rightharpoonup p \quad \text{in} \quad \mathcal{M}(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+ \times [0, T]) - w*, \quad (4.7)$$

$$\bar{p}^\varepsilon \rightharpoonup \bar{p} \quad \text{in} \quad L^\infty(\mathbb{R}_x^d \times V \times [0, T]) - w*, \quad (4.8)$$

$$\bar{p}(x, v, t) = \int_0^\infty p(x, v, m, t) dm, \quad (4.9)$$

$$S^\varepsilon(x, t) \rightarrow S(x, t) \quad \text{strong in} \quad L^q(\mathbb{R}_x^d \times [0, T]), \quad \forall 1 \leq q < \infty, \quad (4.10)$$

and further

$$S^\varepsilon(x, t) \rightarrow S(x, t) \quad \text{in} \quad C_0(\mathbb{R}_x^d \times [0, T]). \quad (4.11)$$

We postpone the proof of Lemma 4.2 to the end of this section. The main theorem in this section is

Theorem 4.3 *Under the assumptions in Lemma 4.2. Let (p, \bar{p}, S) be defined in Lemma 4.2. Then*

$$p(x, v, m, t) = \bar{p}(x, v, t) \delta(m - m_0(S)), \quad (4.12)$$

where $m_0(S)$ is defined in (1.6) such that $F(m_0(S), S) = 0$. And (\bar{p}, S) satisfies, in the sense of distribution,

$$\partial_t \bar{p} + v \cdot \nabla_x \bar{p} = \int_V T(v, v', m_0(S)) \bar{p}(x, v', t) - T(v', v, m_0(S)) \bar{p}(x, v, t) dv', \quad (4.13)$$

$$-\Delta S + S = n(x, t) := \int_V \bar{p}(x, v, t) dv. \quad (4.14)$$

The initial data of the resulting limit equation (4.13) is given by $\bar{p}_0 = \int_0^\infty p_0(x, v, m) dm$.

Proof. Multiply Equation (4.1)₁ by ε to get

$$\partial_m [F(m, S^\varepsilon) p^\varepsilon] = \varepsilon \int_V [T(v, v', m) p^\varepsilon(x, v', m, t) - T(v', v, m) p^\varepsilon(x, v, m, t)] dv' - \varepsilon \partial_t p^\varepsilon - \varepsilon v \cdot \nabla_x p^\varepsilon.$$

Recall we had (4.7) that $p^\varepsilon \rightharpoonup p$ weak* in $\mathcal{M}(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+ \times [0, T])$, and (4.11) that $S^\varepsilon(x, t) \rightarrow S(x, t)$ in $C_0(\mathbb{R}_x^d \times [0, T])$, therefore we can pass to the limit in weak sense as $\varepsilon \rightarrow 0$, up to a subsequence, to get

$$\partial_m [F(m, S) p] = 0 \quad \text{in the sense of measure}, \quad (4.15)$$

thus

$$F(m, S) p \equiv \text{constant} \equiv 0 \quad \text{in the sense of measure.}$$

Using the assumption (1.6) on F , also using the fact (4.9), we readily conclude (4.12).

On the other hand, take the integration of (4.1)₁ with respect to m we have

$$\partial_t \bar{p}^\varepsilon + v \cdot \nabla_x \bar{p}^\varepsilon = \int_0^\infty \int_V T(v, v', m) p^\varepsilon(x, v', m, t) - T(v', v, m) p^\varepsilon(x, v, m, t) dv' dm.$$

By (4.8), we pass to the limit as $\varepsilon \rightarrow 0$ up to a subsequence, and find, in the sense of distribution,

$$\partial_t \bar{p} + v \cdot \nabla_x \bar{p} = \int_0^\infty \int_V T(v, v', m) p(x, v', m, t) - T(v', v, m) p(x, v, m, t) dv' dm.$$

Recall that p has the form in (4.12), then the equation of \bar{p} is thus (4.13).

Next, (4.14) is derived by passing to the limit as $\varepsilon \rightarrow 0$ up to a subsequence on (4.1)₂, using the property (4.10). \square

Proof. *Proof of Lemma 4.2.*

1. Recall the a priori estimate, $\{p^\varepsilon\}$ is bounded in $L^1(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+ \times [0, T])$, thus w^* precompact in $\mathcal{M}(\mathbb{R}_x^d \times V \times \mathbb{R}_m^+ \times [0, T])$, so we have (4.7).

2. Similarly, recall (4.6), $\{\bar{p}^\varepsilon\}$ is uniformly (with respect to ε) bounded in $L^\infty(\mathbb{R}_x^d \times V \times [0, T])$ thus w^* precompact so we have (4.8).

3. To prove (4.9), we have to control the tail for large m . We notice from the assumption (1.6) that

$$F(m, S^\varepsilon) < 0 \quad \text{when} \quad m > m_+,$$

where m_+ is a uniform upper bound of $m_0(S^\varepsilon)$. Now we take a smooth non-decreasing function $\varphi(m)$ such that

$$\varphi(m) = 0 \quad \text{when} \quad m < m_+, \quad \varphi(m) \leq m \quad \text{when} \quad m_+ \leq m \leq 2m_+, \quad \varphi(m) = m \quad \text{when} \quad m > 2m_+,$$

multiply with the equation (4.1)₁, and integrate with respect to x, v, m , then we have

$$\frac{d}{dt} \int_{m_+}^\infty \int_{\mathbb{R}_x^d} \int_V \varphi(m) p^\varepsilon dv dx dm = \int_{m_+}^\infty \int_{\mathbb{R}_x^d} \int_V \varphi'(m) \frac{F(m, S^\varepsilon)}{\varepsilon} p^\varepsilon dv dx dm < 0,$$

thus

$$\begin{aligned} \int_{2m_+}^\infty \int_{\mathbb{R}_x^d} \int_V m p^\varepsilon dv dx dm &\leq \int_{m_+}^\infty \int_{\mathbb{R}_x^d} \int_V \varphi(m) p^\varepsilon dv dx dm \leq \int_{m_+}^\infty \int_{\mathbb{R}_x^d} \int_V \varphi(m) p_0 dv dx dm \\ &\leq \int_0^\infty \int_{\mathbb{R}_x^d} \int_V m p_0 dv dx dm < \infty. \end{aligned} \tag{4.16}$$

The last inequality is based on the assumption (1.7) and gives the control at ∞ . Now choose a test function $\phi(x, v, t) \in C_0$ and a smooth cutoff function

$$0 \leq \chi_R(m) \leq 1, \quad \chi_R(m) = 1 \quad \text{for} \quad m < R, \quad \chi_R(m) = 0 \quad \text{for} \quad m > 2R.$$

Then we compute

$$\begin{aligned} \int_0^T \int_{\mathbb{R}_x^d} \int_V \phi(x, v, t) \bar{p}^\varepsilon dv dx dt &= \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) p^\varepsilon dm dv dx dt \\ &= \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) \chi_R(m) p^\varepsilon dm dv dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) (1 - \chi_R(m)) p^\varepsilon dm dv dx dt. \end{aligned} \tag{4.17}$$

For the first term on the right hand side, note from (4.7) that $p^\varepsilon \rightharpoonup p$ weak*, for any given R ,

$$\int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) \chi_R(m) p^\varepsilon dm dv dx dt \rightarrow \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) \chi_R(m) p dm dv dx dt \quad \text{as } \varepsilon \rightarrow 0.$$

For the second term, the control at ∞ estimate (4.16) ensures

$$\begin{aligned} \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) (1 - \chi_R(m)) p^\varepsilon dm dv dx dt &\leq \|\phi\|_{C_0} \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_R^\infty p^\varepsilon dm dv dx dt \\ &\leq \frac{1}{R} \|\phi\|_{C_0} \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_R^\infty m p^\varepsilon dm dv dx dt \rightarrow 0 \text{ as } R \rightarrow \infty, \end{aligned}$$

thus let $R \rightarrow \infty$ in (4.17) we have

$$\int_0^T \int_{\mathbb{R}_x^d} \int_V \phi(x, v, t) \bar{p}^\varepsilon dv dx dt \rightarrow \int_0^T \int_{\mathbb{R}_x^d} \int_V \int_0^\infty \phi(x, v, t) p dm dv dx dt \quad \text{as } \varepsilon \rightarrow 0.$$

Since we already have (4.8), by uniqueness of the limit we proved (4.9) in the sense of measure.

4. For $\{S^\varepsilon(x, t)\}$, recall (4.4), it is bounded in $L^1 \cap L^\infty(\mathbb{R}_x^d \times [0, T])$ uniformly with respect to ε , to show strong compactness, we use the claims

- Local compactness in space:

$$S^\varepsilon(\cdot, t) \in W^{2,q}(\mathbb{R}_x^d), \quad \forall t \in [0, T], \quad \forall 1 < q < \infty, \quad (4.18)$$

- Local compactness in time:

$$\partial_t S^\varepsilon(\cdot, t) \in W^{1,q}(\mathbb{R}_x^d), \quad \forall t \in [0, T], \quad \forall 1 < q < \infty, \quad (4.19)$$

- Control at ∞ :

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^q dx < \infty, \quad \forall t \in [0, T], \quad \forall 1 \leq q < \infty. \quad (4.20)$$

The first two claims are elliptic estimates which can be similarly derived as in last section. We only prove (4.20). Recall the equation for p^ε :

$$\partial_t p^\varepsilon + v \cdot \nabla_x p^\varepsilon + \partial_m \left[\frac{F(m, S^\varepsilon)}{\varepsilon} p^\varepsilon \right] = \int_V T(v, v', m) p^\varepsilon(x, v', m, t) - T(v', v, m) p^\varepsilon(x, v, m, t) dv'.$$

Multiply both sides of the equation by $\langle x \rangle$ and integrate with respect to x, v, m , we have

$$\frac{d}{dt} \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \langle x \rangle p^\varepsilon dv dm dx = \int_{\mathbb{R}_x^d} \int_0^\infty \int_V \frac{v \cdot x}{\langle x \rangle} p^\varepsilon dv dm dx \leq \|p_0\|_{L^1},$$

that is,

$$\frac{d}{dt} \int_{\mathbb{R}_x^d} \langle x \rangle n^\varepsilon dx \leq \|p_0\|_{L^1},$$

thus we obtain

$$\int_{\mathbb{R}_x^d} \langle x \rangle n^\varepsilon dx \quad \text{is uniformly bounded for all } t \in [0, T]. \quad (4.21)$$

Next, recall the equation for S^ε :

$$-\Delta S^\varepsilon + S^\varepsilon = n^\varepsilon,$$

multiply both sides of the equation by $\langle x \rangle (S^\varepsilon)^{q-1}$, for any q large and fixed, then integrate with respect to x , we have

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^{q-1} (-\Delta S^\varepsilon + S^\varepsilon) dx = \int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^{q-1} n^\varepsilon dx,$$

then

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^q dx + \int_{\mathbb{R}_x^d} \nabla(\langle x \rangle (S^\varepsilon)^{q-1}) \cdot \nabla S^\varepsilon dx = \int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^{q-1} n^\varepsilon dx.$$

Note here that

$$\begin{aligned} \int_{\mathbb{R}_x^d} \nabla(\langle x \rangle (S^\varepsilon)^{q-1}) \cdot \nabla S^\varepsilon dx &= \int_{\mathbb{R}_x^d} \langle x \rangle (p-1) (S^\varepsilon)^{q-2} \nabla S^\varepsilon \cdot \nabla S^\varepsilon + \int_{\mathbb{R}_x^d} (S^\varepsilon)^{q-1} \nabla S^\varepsilon \cdot \nabla \langle x \rangle \\ &= \int_{\mathbb{R}_x^d} \langle x \rangle (p-1) (S^\varepsilon)^{q-2} |\nabla S^\varepsilon|^2 - \int_{\mathbb{R}_x^d} \frac{1}{q} (S^\varepsilon)^q \Delta \langle x \rangle \\ &= \int_{\mathbb{R}_x^d} \langle x \rangle (p-1) (S^\varepsilon)^{q-2} |\nabla S^\varepsilon|^2 - \int_{\mathbb{R}_x^d} \frac{1}{q} (S^\varepsilon)^q \left(\frac{d-1}{\langle x \rangle} + \frac{1}{\langle x \rangle^3} \right). \end{aligned}$$

Then, we find

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^q dx + \int_{\mathbb{R}_x^d} \langle x \rangle (q-1) (S^\varepsilon)^{q-2} |\nabla S^\varepsilon|^2 dx = \int_{\mathbb{R}_x^d} \frac{1}{q} (S^\varepsilon)^q \left(\frac{d-1}{\langle x \rangle} + \frac{1}{\langle x \rangle^3} \right) dx + \int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^{q-1} n^\varepsilon dx,$$

that is

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^q dx \leq \frac{d}{q} \int_{\mathbb{R}_x^d} (S^\varepsilon)^q dx + \|S^\varepsilon\|_{L_x^\infty}^{q-1} \int_{\mathbb{R}_x^d} \langle x \rangle n^\varepsilon dx,$$

using again (4.4) that $\{S^\varepsilon(\cdot, t)\}$ is uniformly bounded in $L_x^1 \cap L_x^\infty$ for all $t \in [0, T]$, and the fact (4.21), we conclude, for any fixed q large,

$$\int_{\mathbb{R}_x^d} \langle x \rangle (S^\varepsilon)^q dx < \infty, \quad \forall t \in [0, T].$$

Note also (4.20) already holds for $q = 1$, which is because we can use the similar argument as for (2.16), therefore we have proved (4.20) for all $1 \leq q < \infty$.

In conclusion, by using again the strong compactness criterion in Brezis [8], we have (4.10) from the above claims.

5. To prove (4.11), we first estimate the control of S^ε at $x = \infty$. For simplicity of notation, we abbreviate the time variable in the following.

Fix $R > 1$. For $|x| \geq R$, we compute

$$\begin{aligned} |x|^{1/d} S^\varepsilon(x) &= \int |x|^{1/d} G(x-y) n^\varepsilon(y) dy \\ &\leq \int |x-y|^{1/d} G(x-y) n^\varepsilon(y) dy + \int |y|^{1/d} G(x-y) n^\varepsilon(y) dy \\ &=: I^\varepsilon + II^\varepsilon. \end{aligned}$$

For I^ε , note that G is only singular at the origin, we cutoff the singularity and estimate

$$\begin{aligned}
I^\varepsilon &= \int_{|x-y|<1} |x-y|^{1/d} G(x-y) n^\varepsilon(y) dy + \int_{|x-y|>1} |x-y|^{1/d} G(x-y) n^\varepsilon(y) dy \\
&\leq \int_{|x-y|<1} G(x-y) n^\varepsilon(y) dy + \left(\int_{|x-y|>1} |x-y|^{2/d} G^2(x-y) dy \right)^{\frac{1}{2}} \|n^\varepsilon\|_{L_x^2} \\
&\leq \|n^\varepsilon\|_{L_x^\infty} + \left(\int_{|x|>1} |x|^{2/d} G^2(x) dx \right)^{\frac{1}{2}} \|n^\varepsilon\|_{L_x^2},
\end{aligned}$$

recall (4.5), note also the L^1 bound of $\{n^\varepsilon\}$ is trivial from conservation of mass, thus $\{n^\varepsilon\}$ is uniformly bounded in $L_x^1 \cap L_x^\infty$, and note the property (P3) of Bessel potential $G(x)$ in Proposition 2.1, the integration above is finite. Then we conclude that I^ε is uniformly bounded (the bound is independent of ε).

For II^ε , we use Hölder's inequality, with $q = \frac{d}{d-1}$ and $q' = d$, to get

$$\begin{aligned}
II^\varepsilon &= \int |y|^{1/d} G(x-y) n^\varepsilon(y) dy \leq \|G\|_{L^{\frac{d}{d-1}}} \left(\int |y| n^\varepsilon(y)^d dy \right)^{\frac{1}{d}} \\
&\leq \|G\|_{L^{\frac{d}{d-1}}} \left(\|n^\varepsilon\|_{L_x^\infty}^{d-1} \int |y| n^\varepsilon(y) dy \right)^{\frac{1}{d}}.
\end{aligned} \tag{4.22}$$

Here we use the property of Bessel potential (P2) in Proposition 2.1, together with the estimates (4.5) and (4.21), we conclude from (4.22) that II^ε is also uniformly bounded (the bound is independent of ε). Combine the above, we conclude

$$|x|^{1/d} S^\varepsilon(x) < \infty \quad \text{for all } |x| > R (> 1). \tag{4.23}$$

Next, by Morrey type Sobolev embedding Theorem (see Evans [17]), note the elliptic estimates (4.18)-(4.19) imply that the convergence of $S^\varepsilon(x, t) \rightarrow S(x, t)$ is actually in Hölder space $C^{0,1-\delta}(\mathbb{R}_x^d \times [0, T])$, for any $\delta \in (0, 1)$. Together with the estimates (4.23), which yields that $S(x)$ vanishes at infinity, then (4.11) is proved. \square

5 Conclusion

We have considered a nonlinear kinetic chemotaxis model with internal dynamics incorporating signal transduction and adaptation. Under some quite general assumptions on the model and the initial data, we have proved the global existence of weak solution by using the Schauder fixed point theorem. More precisely, for our mathematical treatment, we generalise the assumptions on the adaptation rate F and turning kernel T in the model than that in [14, 15, 16]. Compare with the global existence result in [14], our result holds for any physical space dimensions. Moreover, the uniqueness of weak solution is also derived, based on some further regularity estimates on the solutions, following the method devised in [26].

Next, we considered the fast adaptation limit of this model to Othmer-Dunbar-Alt type kinetic chemotaxis model. This limit gives some insight to the molecular origin of the chemotaxis behaviour, by incorporating information about microscopic intracellular processes such as signal transduction and response into the chemotaxis description. This was done in [15, 16] for a highly simplified description of intracellular dynamics, where linear dynamics for the response to an extracellular signal was assumed.

We remark also that in order to derive the molecular origin of the chemotaxis behaviour, we did not use any moment closure to derive the closed evolution equation for the macroscopic density of cells as in [15, 16], instead, a kinetic type limit equation (4.13) is arrived, with turning kernel incorporating information from microscopic intracellular processes.

In our analysis, the fast adaptation limit is derived by extracting a weak convergence subsequence in measure space. For this limit, the first difficulty is to show the concentration effect on the internal state. When the small parameter ε , the adaptation time scale, goes to zero, we prove that the solution converges to a Dirac mass in the internal state variable. Another difficulty is the strong compactness argument on the chemical potential, which is essential for passing the nonlinear kinetic equation to the weak limit.

For future works, it is interesting to consider the case with several chemical reactions as related to the model proposed in Erban-Othmer [15, 16]. We also mention that other types of scaling and related limits can be considered. For example, based on both fast adaptation and stiff response, the paper [30] studied how the path-wise gradient of chemotactic signal arises from intra-cellular molecular content. Several other rescallings are possible. For instance, an open problem is the asymptotic behaviour of the hyperbolic scaling, with $(1.1)_1$ replaced by

$$\partial_t p + v \cdot \nabla_x p + \partial_m \left[\frac{F(m, S)}{\varepsilon} p \right] = \frac{1}{\varepsilon} \int_V T(v, v', m) p(x, v', m, t) - T(v', v, m) p(x, v, m, t) dv'.$$

The hyperbolic limit is quite different from the fast adaptation limit, because, in that case, even the a priori estimate for \bar{p} is not uniform in ε , so the limit equation is unclear, a difficult part of that will be to determine how the extracellular signal feeds into the chemotaxis response of the cells.

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